Computational Fluid Dynamics - An Introduction

Lecture Notes

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Contents

Chapter 1

Introduction

Chapter 2

The Basic Hydrodynamic Equations and Their Mathematical Structure

2.1 The Basic Hydrodynamic Equations

2.1.1 Continuitiy Equation

Non-conservative form

$$\frac{D\rho}{Dt} + \rho\vec{\nabla}\cdot\vec{V} = 0 \tag{2.1}$$

conservative form

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{V}\right) = 0 \tag{2.2}$$

2.1.2 Momentum Equation

Non-conservative form

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\tau_{xx}}{\partial x} + \frac{\tau_{yx}}{\partial y} + \frac{\tau_{zx}}{\partial z} + \rho f_x \tag{2.3}$$

$$\rho\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\tau_{xy}}{\partial x} + \frac{\tau_{yy}}{\partial y} + \frac{\tau_{zy}}{\partial z} + \rho f_y \tag{2.4}$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\tau_{xz}}{\partial x} + \frac{\tau_{yz}}{\partial y} + \frac{\tau_{zz}}{\partial z} + \rho f_z \tag{2.5}$$

conservative form

$$\rho\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\left(\rho u\vec{V}\right) = -\frac{\partial p}{\partial x} + \frac{\tau_{xx}}{\partial x} + \frac{\tau_{yx}}{\partial y} + \frac{\tau_{zx}}{\partial z} + \rho f_x \tag{2.6}$$

$$\rho\frac{\partial v}{\partial t} + \vec{\nabla}\cdot\left(\rho u\vec{V}\right) = -\frac{\partial p}{\partial y} + \frac{\tau_{xy}}{\partial x} + \frac{\tau_{yy}}{\partial y} + \frac{\tau_{zy}}{\partial z} + \rho f_y \tag{2.7}$$

$$\rho\frac{\partial w}{\partial t}+\vec{\nabla}\cdot\left(\rho u\vec{V}\right)=-\frac{\partial p}{\partial z}+\frac{\tau_{xz}}{\partial x}+\frac{\tau_{yz}}{\partial y}+\frac{\tau_{zz}}{\partial z}+\rho f_z \tag{2.8}$$

2.1.3 Energy Equation

Non-conservative form

$$\begin{aligned}\rho\frac{D}{Dt}\left(e+\frac{V^2}{2}\right) &= \rho\dot{q}+\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\\ &-\frac{\partial(up)}{\partial x}-\frac{\partial(vp)}{\partial y}-\frac{\partial(wp)}{\partial z}\\ &+\frac{u\tau_{xx}}{\partial x}+\frac{u\tau_{yx}}{\partial y}+\frac{u\tau_{zx}}{\partial z}\\ &+\frac{u\tau_{xy}}{\partial x}+\frac{v\tau_{yy}}{\partial y}+\frac{v\tau_{zy}}{\partial z}\\ &+\frac{w\tau_{xz}}{\partial x}+\frac{w\tau_{yz}}{\partial y}+\frac{w\tau_{zz}}{\partial z}+\rho\vec{f}\cdot\vec{V}\end{aligned} \tag{2.9}$$

conservative form

$$\begin{aligned}\rho\frac{\partial}{\partial t}\left(e+\frac{V^2}{2}\right) &+ \vec{\nabla}\cdot\left[\rho\left(e+\frac{V^2}{2}\right)\vec{V}\right]\\ &= \rho\dot{q}+\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\\ &-\frac{\partial(up)}{\partial x}-\frac{\partial(vp)}{\partial y}-\frac{\partial(wp)}{\partial z}\\ &+\frac{u\tau_{xx}}{\partial x}+\frac{u\tau_{yx}}{\partial y}+\frac{u\tau_{zx}}{\partial z}\\ &+\frac{u\tau_{xy}}{\partial x}+\frac{v\tau_{yy}}{\partial y}+\frac{v\tau_{zy}}{\partial z}\\ &+\frac{w\tau_{xz}}{\partial x}+\frac{w\tau_{yz}}{\partial y}+\frac{w\tau_{zz}}{\partial z}+\rho\vec{f}\cdot\vec{V}\end{aligned} \tag{2.10}$$

2.2 The Euler Equations

2.2.1 Continuitiy Equation

Non-conservative form

$$\frac{D\rho}{Dt}+\rho\vec{\nabla}\cdot\vec{V}=0 \tag{2.11}$$

conservative form

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot\left(\rho\vec{V}\right)=0 \tag{2.12}$$

2.2.2 Momentum Equation

Non-conservative form

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho f_x \tag{2.13}$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \rho f_y \tag{2.14}$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \rho f_z \tag{2.15}$$

conservative form

$$\rho \frac{\partial u}{\partial t} + \vec{\nabla} \cdot \left(\rho u \vec{V}\right) = -\frac{\partial p}{\partial x} + \rho f_x \tag{2.16}$$

$$\rho \frac{\partial v}{\partial t} + \vec{\nabla} \cdot \left(\rho u \vec{V}\right) = -\frac{\partial p}{\partial y} + \rho f_y \tag{2.17}$$

$$\rho \frac{\partial w}{\partial t} + \vec{\nabla} \cdot \left(\rho u \vec{V}\right) = -\frac{\partial p}{\partial z} + \rho f_z \tag{2.18}$$

2.2.3 Energy Equation

Non-conservative form

$$\rho \frac{D}{Dt}\left(e + \frac{V^2}{2}\right) = \rho \dot{q} + \tag{2.19}$$

$$-\frac{\partial (up)}{\partial x} - \frac{\partial (vp)}{\partial y} - \frac{\partial (wp)}{\partial z} + \rho \vec{f} \cdot \vec{V} \tag{2.20}$$

conservative form

$$\rho \frac{\partial}{\partial t}\left(e + \frac{V^2}{2}\right) + \vec{\nabla} \cdot \left[\rho \left(e + \frac{V^2}{2}\right) \vec{V}\right]$$

$$= \rho \dot{q} + -\frac{\partial (up)}{\partial x} - \frac{\partial (vp)}{\partial y} - \frac{\partial (wp)}{\partial z} + \rho \vec{f} \cdot \vec{V} \tag{2.21}$$

2.3 Mathematical Structure

Chapter 3

Partial Differential Equations

Chapter 4

Grids - a course in coordinate transformations

Chapter 5

Numerical Techniques for PDEs

Chapter 6

First Example: The Flow Through a Nozzle

We will use MacCormack's technique to solve the subsonic-supersonic isentropic inviscid flow through a nozzle.

6.1 MacCormack's Technique

MacCormack's technique is an easily implemented time-marching predictor-corrector scheme for hyperbolic PDEs. Here, we show the implementation of a one-dimensional (in space) code.

6.1.1 Predictor Step

We use the continuity, momentum and energy equations in dimensionless form to obtain the following three *forward difference* equations for the time derivatives of density, speed, and temperature:

$$\left(\frac{\partial \rho}{\partial t}\right)_i^t = -\rho_i^t \frac{v_{i+1}^t - v_i^t}{\Delta x} - \rho_i^t v_i^t \frac{\ln A_{i+1} - \ln A_i}{\Delta x} - v_i^t \frac{\rho_{i+1}^t - \rho_i^t}{\Delta x} \tag{6.1}$$

$$\left(\frac{\partial v}{\partial t}\right)_i^t = -v_i^t \frac{v_{i+1}^t - v_i^t}{\Delta x} - \frac{1}{\gamma}\left(\frac{T_{i+1}^t - T_i^t}{\Delta x} + \frac{T_i^t}{\rho_i^t}\frac{\rho_{i+1}^t - \rho_i^t}{\Delta x}\right) \tag{6.2}$$

$$\begin{aligned}\left(\frac{\partial T}{\partial t}\right)_i^t &= -v_i^t \frac{T_{i+1}^t - T_i^t}{\Delta x} \\ &\quad -(\gamma - 1)T_i^t \left(\frac{v_{i+1}^t - v_i^t}{\Delta x} + v_i^t \frac{\ln A_{i+1} - \ln A_i}{\Delta x}\right)\end{aligned} \tag{6.3}$$

Using these derivatives, we obtain predicted values for density, speed, and temperature:

$$\bar{\rho}_i^{t+\Delta t} \doteq \rho_i^t + \left(\frac{\partial \rho}{\partial t}\right)_i^t \Delta t \tag{6.4}$$

$$\bar{v}_i^{t+\Delta t} \doteq v_i^t + \left(\frac{\partial v}{\partial t}\right)_i^t \Delta t \tag{6.5}$$

$$\bar{T}_i^{t+\Delta t} \doteq T_i^t + \left(\frac{\partial T}{\partial t}\right)_i^t \Delta t \tag{6.6}$$

This completes the predictor step.

6.1.2 Corrector Step

Using the *predicted* values for density, speed, and temperature, derive the *corrector* derivatives:

$$\begin{aligned}
\left(\frac{\partial \bar{\rho}}{\partial t}\right)_i^t &= -\bar{\rho}_i^{t+\Delta t}\frac{\bar{v}_{i+1}^{t+\Delta t} - \bar{v}_i^{t+\Delta t}}{\Delta x} - \bar{\rho}_i^{t+\Delta t}\bar{v}_i^{t+\Delta t}\frac{\ln A_{i+1} - \ln A_i}{\Delta x} \\
&\quad -\bar{v}_i^{t+\Delta t}\frac{\bar{\rho}_{i+1}^{t+\Delta t} - \bar{\rho}_i^{t+\Delta t}}{\Delta x} &(6.7)\\
\left(\frac{\partial \bar{v}}{\partial t}\right)_i^t &= -\bar{v}_i^{t+\Delta t}\frac{\bar{v}_{i+1}^{t+\Delta t} - \bar{v}_i^{t+\Delta t}}{\Delta x} \\
&\quad -\frac{1}{\gamma}\left(\frac{\bar{T}_{i+1}^{t+\Delta t} - \bar{T}_i^{t+\Delta t}}{\Delta x} + \frac{\bar{T}_i^{t+\Delta t}}{\bar{\rho}_i^{t+\Delta t}}\frac{\bar{\rho}_{i+1}^{t+\Delta t} - \bar{\rho}_i^{t+\Delta t}}{\Delta x}\right) &(6.8)\\
\left(\frac{\partial \bar{T}}{\partial t}\right)_i^t &= -\bar{v}_i^{t+\Delta t}\frac{\bar{T}_{i+1}^{t+\Delta t} - \bar{T}_i^{t+\Delta t}}{\Delta x} \\
&\quad -(\gamma-1)\bar{T}_i^{t+\Delta t}\left(\frac{\bar{v}_{i+1}^{t+\Delta t} - \bar{v}_i^{t+\Delta t}}{\Delta x} + \bar{v}_i^{t+\Delta t}\frac{\ln A_{i+1} - \ln A_i}{\Delta x}\right) &(6.9)
\end{aligned}$$

We now define derivatives which are the average of the predictor and corrector derivatives:

$$\left(\frac{\partial \rho}{\partial t}\right)_{\text{ave}} \doteq \frac{1}{2}\left[\left(\frac{\partial \rho}{\partial t}\right)_i^t + \left(\frac{\partial \bar{\rho}}{\partial t}\right)_i^t\right] \tag{6.10}$$

$$\left(\frac{\partial v}{\partial t}\right)_{\text{ave}} \doteq \frac{1}{2}\left[\left(\frac{\partial v}{\partial t}\right)_i^t + \left(\frac{\partial \bar{v}}{\partial t}\right)_i^t\right] \tag{6.11}$$

$$\left(\frac{\partial T}{\partial t}\right)_{\text{ave}} \doteq \frac{1}{2}\left[\left(\frac{\partial T}{\partial t}\right)_i^t + \left(\frac{\partial \bar{T}}{\partial t}\right)_i^t\right] \tag{6.12}$$

Finally, using these averaged derivatives, we determine the corrected values of the flow-field variables:

$$\rho_i^{t+\Delta t} = \rho_i^t + \left(\frac{\partial \rho}{\partial t}\right)_{\text{ave}} \Delta t \tag{6.13}$$

$$v_i^{t+\Delta t} = v_i^t + \left(\frac{\partial v}{\partial t}\right)_{\text{ave}} \Delta t \tag{6.14}$$

$$T_i^{t+\Delta t} = T_i^t + \left(\frac{\partial T}{\partial t}\right)_{\text{ave}} \Delta t \tag{6.15}$$

This completes the corrector step. We now have the flow-field quantities at the next time step.

The full algorithm still needs to account for boundary conditions. This is done in class. These notes only serve as such for the exercise class.

You can use the script `maccormack.py` to play around with the method and change the cross section A of the nozzle. It is in the usual directory:

`www.ieap.uni-kiel.de/et/people/wimmer/teaching/hydro/ex/ex6`

Chapter 7

An Example: Supersonic Flow

Chapter 8

Introduction

Chapter 9

An Example Involving the Navier-Stokes Equations

Appendix A

Some Aspects of python

Appendix B

Plotting Results With Gnuplot and Python

Appendix C

Some Aspects of numpy

Appendix D

Some Aspects of scipy