Chapter 1

Summary of hydrodynamical equations

1.1 Viscous Flows: Navier-Stokes Equations

1.1.1 Continuitiy Equation

Non-conservative form

$$\frac{D\rho}{Dt} + \rho\vec{\nabla}\cdot\vec{V} = 0 \tag{1.1}$$

conservative form

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{V}\right) = 0 \tag{1.2}$$

1.1.2 Momentum Equation

Non-conservative form

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\tau_{xx}}{\partial x} + \frac{\tau_{yx}}{\partial y} + \frac{\tau_{zx}}{\partial z} + \rho f_x \tag{1.3}$$

$$\rho\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\tau_{xy}}{\partial x} + \frac{\tau_{yy}}{\partial y} + \frac{\tau_{zy}}{\partial z} + \rho f_y \tag{1.4}$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\tau_{xz}}{\partial x} + \frac{\tau_{yz}}{\partial y} + \frac{\tau_{zz}}{\partial z} + \rho f_z \tag{1.5}$$

conservative form

$$\rho\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\left(\rho u\vec{V}\right) = -\frac{\partial p}{\partial x} + \frac{\tau_{xx}}{\partial x} + \frac{\tau_{yx}}{\partial y} + \frac{\tau_{zx}}{\partial z} + \rho f_x \tag{1.6}$$

$$\rho\frac{\partial v}{\partial t} + \vec{\nabla}\cdot\left(\rho u\vec{V}\right) = -\frac{\partial p}{\partial y} + \frac{\tau_{xy}}{\partial x} + \frac{\tau_{yy}}{\partial y} + \frac{\tau_{zy}}{\partial z} + \rho f_y \tag{1.7}$$

$$\rho\frac{\partial w}{\partial t} + \vec{\nabla}\cdot\left(\rho u\vec{V}\right) = -\frac{\partial p}{\partial z} + \frac{\tau_{xz}}{\partial x} + \frac{\tau_{yz}}{\partial y} + \frac{\tau_{zz}}{\partial z} + \rho f_z \tag{1.8}$$

1.1.3 Energy Equation

Non-conservative form

$$
\begin{aligned}
\rho \frac{D}{Dt}\left(e + \frac{V^2}{2}\right) = \; & \rho \dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) \\
& - \frac{\partial (up)}{\partial x} - \frac{\partial (vp)}{\partial y} - \frac{\partial (wp)}{\partial z} \\
& + \frac{u\tau_{xx}}{\partial x} + \frac{u\tau_{yx}}{\partial y} + \frac{u\tau_{zx}}{\partial z} \\
& + \frac{u\tau_{xy}}{\partial x} + \frac{v\tau_{yy}}{\partial y} + \frac{v\tau_{zy}}{\partial z} \\
& + \frac{w\tau_{xz}}{\partial x} + \frac{w\tau_{yz}}{\partial y} + \frac{w\tau_{zz}}{\partial z} + \rho \vec{f} \cdot \vec{V} \qquad (1.9)
\end{aligned}
$$

conservative form

$$
\begin{aligned}
\rho \frac{\partial}{\partial t}\left(e + \frac{V^2}{2}\right) \; & + \; \vec{\nabla} \cdot \left[\rho \left(e + \frac{V^2}{2}\right) \vec{V}\right] \\
& = \rho \dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) \\
& \quad - \frac{\partial (up)}{\partial x} - \frac{\partial (vp)}{\partial y} - \frac{\partial (wp)}{\partial z} \\
& \quad + \frac{u\tau_{xx}}{\partial x} + \frac{u\tau_{yx}}{\partial y} + \frac{u\tau_{zx}}{\partial z} \\
& \quad + \frac{u\tau_{xy}}{\partial x} + \frac{v\tau_{yy}}{\partial y} + \frac{v\tau_{zy}}{\partial z} \\
& \quad + \frac{w\tau_{xz}}{\partial x} + \frac{w\tau_{yz}}{\partial y} + \frac{w\tau_{zz}}{\partial z} + \rho \vec{f} \cdot \vec{V} \qquad (1.10)
\end{aligned}
$$

1.2 Inviscid Flows: The Euler Equations

1.2.1 Continuitiy Equation

Non-conservative form

$$\frac{D\rho}{Dt} + \rho\vec{\nabla}\cdot\vec{V} = 0 \tag{1.11}$$

conservative form

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{V}\right) = 0 \tag{1.12}$$

1.2.2 Momentum Equation

Non-conservative form

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho f_x \tag{1.13}$$

$$\rho\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \rho f_y \tag{1.14}$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \rho f_z \tag{1.15}$$

conservative form

$$\rho\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\left(\rho u\vec{V}\right) = -\frac{\partial p}{\partial x} + \rho f_x \tag{1.16}$$

$$\rho\frac{\partial v}{\partial t} + \vec{\nabla}\cdot\left(\rho u\vec{V}\right) = -\frac{\partial p}{\partial y} + \rho f_y \tag{1.17}$$

$$\rho\frac{\partial w}{\partial t} + \vec{\nabla}\cdot\left(\rho u\vec{V}\right) = -\frac{\partial p}{\partial z} + \rho f_z \tag{1.18}$$

1.2.3 Energy Equation

Non-conservative form

$$\rho\frac{D}{Dt}\left(e + \frac{V^2}{2}\right) = \rho\dot{q} + \tag{1.19}$$

$$-\frac{\partial(up)}{\partial x} - \frac{\partial(vp)}{\partial y} - \frac{\partial(wp)}{\partial z} + \rho\vec{f}\cdot\vec{V} \tag{1.20}$$

conservative form

$$\rho\frac{\partial}{\partial t}\left(e + \frac{V^2}{2}\right) + \vec{\nabla}\cdot\left[\rho\left(e + \frac{V^2}{2}\right)\vec{V}\right]$$

$$= \rho\dot{q} + -\frac{\partial(up)}{\partial x} - \frac{\partial(vp)}{\partial y} - \frac{\partial(wp)}{\partial z} + \rho\vec{f}\cdot\vec{V} \tag{1.21}$$