Solve the linear convection equation

$$u_t + au_x = 0$$

using the following schemes:

- first order upwind (FOU)
- Lax-Friedrich
- Lax-Wendroff
- Leapfrog
- Warming and Beam
- MacCormack

which are defined by:
FOU:

$$u_i^{n+1} = u_i^n - s(u_i^n - u_{i-1}^n),$$

Lax-Friedrich:

$$u_i^{n+1} = 1/2(u_{i+1}^n + u_{i-1}^n) - s/2(u_{i+1}^n - u_{i-1}^n),$$

Lax-Wendroff:

$$u_i^{n+1} = u_i^n - s/2(u_{i+1}^n - u_{i-1}^n) + s^2/2(u_{i+1}^n - 2u_i^n + u_{i-1}^n),$$

Leapfrog:

$$u_i^{n+1} = u_i^n + s/2(u_{i+1}^n - u_{i-1}^n)$$

Warming and Beam:

$$u_i^{n+1} = u_i^n - s/2(3u_i^n - 4u_{i-1}^n + u_{i-2}^n) + s^2/2(u_i^n - 2u_{i-1}^n + u_{i-2}^n),$$

and MacCormack's method (this is a predictor-corrector method):

$$\begin{aligned} ub_i^n + 1 &= u_i^n - s * (u_i^n - i_i - 1^n), \\ u_i^n + 1 &= 1/2 * (ub_i^n + 1 + u_i^n) - s/2 * (ub_i + 1^n + 1 - ub_i^n + 1), \end{aligned}$$

where $s = adt/dx$
See p. 298 ff in Hirsch, vol. 1.
This exercise is intended to help understand chapter 7 of Hirsch on 'Consistency, Stability, and Error Analysis'. Most examples are taken from there some have been added, some excluded.
In this exercise we don't treat flux limiters. For that, see ex9 in comp. hydrodynamics

Tasks:

1) Perform a stability analysis for the FOU scheme. You should find that it is conditionally stable for $0 \leq \sigma \leq 1$. Try out the FOU scheme for various values of sigma (which can be adjusted in the global definitions below).

2) Repeat the same for the leapfrog scheme. You should find that it is neutrally stable: $|G| = 1$ for $|\sigma| \leq 1$.

3) Repeat the same for the Lax-Wendroff scheme. You should find that it is stable for $|\sigma| \leq 1$.

4) Using CFL = 0.8 plot the results of all schemes after the same number of steps, e.g., 100 time steps. Compare.

5) Experiment with different numbers of spatial points. Compare FOU and Lax-Friedrichs for small $N \sim 100$. You will see the decoupling of the even and odd numbers which is inherent in Lax-Friedrichs but not in FOU.

6) Experiment with various combinations of values for the wave number, k, CFL numbers, σ, and number of mesh points per wave length. This influences the phase value, $\varphi = k\Delta x$. Compare with your stability analysis of the various schemes. You should now be able to understand why the amplitude of the wave solutions of some schemes decreases with time. What happens if you increase the number of mesh points per wavelength? Where does that put you in a plot of Diffusion error vs. phase angle?

7) The group velocity of a signal is given by $v_g \doteq \frac{\mathrm{d}\omega}{\mathrm{dk}}$. Determine the growth factor G for the leapfrog scheme. You should get

$$G = \exp\left(-i\omega\Delta t\right) = \cos\omega\Delta t - i\sin\omega\Delta t = -i\sigma\sin\phi \pm \sqrt{1 - \sigma^2\sin^2\phi}.$$

Identifying the imagninary parts you obtain

$$\sin\omega\Delta t = \sigma\sin\phi,$$

from which we obtain with $\phi = k\Delta x$ and $\sigma = a\Delta t/\Delta x$

$$v_g(k) = a\frac{\cos\phi}{\cos\omega\Delta t} = a\frac{\cos\phi}{\sqrt{1 - \sigma^2\sin^2\phi}}.$$

Plot the group velocity and check the behaviour of the leapfrog scheme against the expected one. You should find that, indeed, for a CFL of 0.4 and $\phi = k\Delta x = \pi/4$, the Gaussian wave packet moves at about 3/4 of the real speed. This effect is entirely numerical in nature! The high frequency part of the numerical solution also propagates to the left and is reflected by the left-hand boundary, as can be seen by inserting $\phi \sim \pi$ into the above expression for the group velocity.